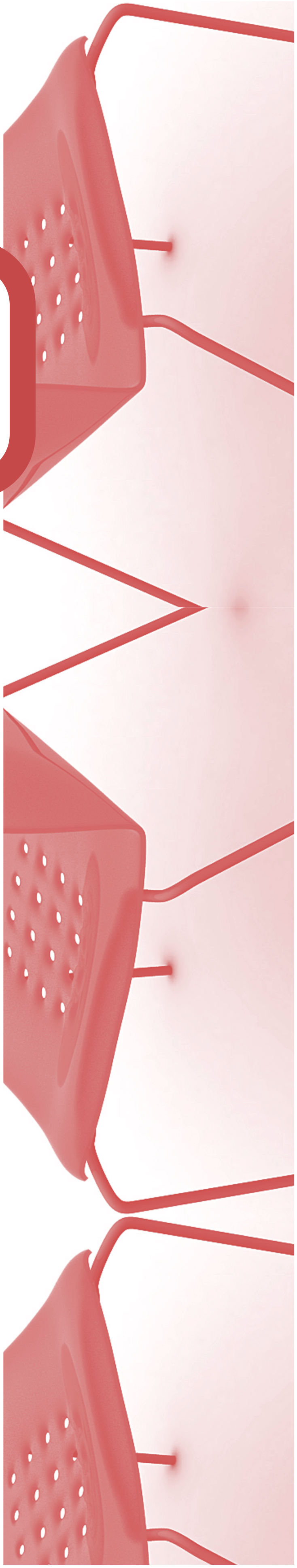
PART VII

GUIDE TO USEFUL MATHEMATICAL METHODS

Introduction

Part 7 differs significantly from the other parts of the book, since it deals with a certainty which can exist in mathematical relationships but which does not arise within statistical relationships. It is also different in the way in which it deals with the topics. Because of the ‘theoretical’ nature of this subject matter we have decided not to include case studies for this section, but you will find that we draw a large number of our examples from economics and economic theory. The reason for moving materials to the web is that many courses do not now cover this area in depth, but we wish to continue to provide a fully comprehensive coverage for those who need these materials.

You may already have read through and used Chapter 18 earlier in this book, and it is the intention of this Part to build on that basic foundation. Mathematics offers a concise, exact method of describing a situation together with a set of analysis tools which are well proven and well used. Such analysis allows different perspectives to be explored and new relationships to be determined. These can then be tested in the practical context by the use of the statistical techniques which have been introduced in the earlier Parts of this book. Subjects such as microeconomic theory can be developed almost wholly using mathematics, and if your course uses mathematics extensively in this way, then this part will give you the background that you need.



25

A GUIDE TO MATRICES

The algebra used in Chapter 18 allows the specification and solution of a range of business problems and provides the framework for the theoretical development of disciplines such as economics. We have seen single equations solved in terms of x and simultaneous equations solved in terms of x and y. However, to develop more complex theoretical models we must be able to deal effectively with problems involving more than two equations. **Matrix** notation provides a way of describing these more complex problems and the rules of matrices provide a way of manipulating these problems. It must be remembered that matrix algebra only provides a convenient notation and some new methods of solution. Matrices will not solve problems that are not amenable to solution by other methods. As with the application of all quantitative methods, problem specification is the important first step.

Matrices are particularly useful in solving sets of simultaneous equations. The compact form of notation is consistent with the use of arrays in computer programming, and many computer packages will offer the facility of matrix manipulation, e.g. MINITAB. Matrices are also used extensively in multiple regression (Chapter 16), and the methods of matrix algebra in more advanced work on linear programming (Chapter 20). Further applications include probability modelling, such as Markov Chains, section 25.4.

Objectives

After working through this chapter, you should be able to:

- define a matrix
- add and subtract matrices
- multiply matrices
- invert square matrices

- solve simultaneous equations using matrices
- solve Markov chain problems.
- solve input/output problems.

25.1 What is a matrix?

A matrix is a *rectangular array* of numbers arranged in *rows* and *columns* and is characterized by its size (or *order*), written as (no. of rows × no. of columns). The whole matrix is usually referred to by a capital letter, while individual numbers, or *elements*, within the matrix are referred to by lower case letters, usually with a suffix to identify in which row and in which column they appear. Note that a matrix does not have a numerical value; it is merely a convenient way of representing an array of numbers. If

$$A = \begin{bmatrix} 4 & 8 & 17 & 12 \\ 21 & 3 & 19 & 17 \\ 10 & 21 & 4 & 2 \end{bmatrix}$$

then the order of matrix **A** is (3 × 4) and the element a_{13} is the 17 since it is in the first row and the third column.

It is often convenient to use the double subscript notation where a_{ij} denotes the element located in the ith row and jth column of matrix **A** and b_{ij} denotes the element located in the ith row and jth column of matrix **B**.

If a matrix has only one row, then it is known as a *row vector*; if it has only one column, then it is a *column vector*, e.g.

$$B = [4 \quad 8 \quad 7] \qquad C = \begin{bmatrix} 10 \\ 12 \\ 28 \\ 49 \\ 102 \end{bmatrix}$$

The order of matrix **B** is (1 × 3) and the order of matrix **C** is (5 × 1).

EXAMPLE

A company, Comfy Chairs Ltd, produces three types of chair, the Classic, the Victorian and the Modern in its existing factory. Weekly output of the Classic, the Victorian and the Modern types in mahogany are 30, 60 and 80 respectively, and in teak, 20, 30 and 40 respectively.

This information can easily be represented by a matrix, say **D**, where columns refer to chair type and the rows to the wood used.

$$D = \begin{bmatrix} 30 & 60 & 80 \\ 20 & 30 & 40 \end{bmatrix}$$

25.2 Matrix manipulation

We will find that many sets of data and many situations can be described in terms of rows and columns. In this section we consider the rules governing matrix algebra that will allow us to manipulate these rows and columns.

25.2.1 Equality of matrices

The concept of equality is fundamental to all algebra. In matrix algebra, two matrices **A** and **B** are equal only if they are of the same size *and* their corresponding elements are equal, $a_{ij} = b_{ij}$ for *all* values of i and j.

EXAMPLE

Matrix **E** is only equal to matrix **D** if matrix **E** is of the same size, (2 × 3), and each element of matrix **E** is equal to each element of matrix **D** (above). If

$$D = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

then $a = 30$, $b = 60$, $c = 80$, $d = 20$, $e = 30$ and $f = 40$.

25.2.2 Addition and subtraction of matrices

To add or subtract matrices they must be of the same order; they are then said to be *conformable for addition*. When this is true, for addition the corresponding elements in each matrix are added together and for subtraction, each of the second elements is subtracted from the corresponding elements in the first. If **A** and **B** are two matrices of the same size, then $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$ and $\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}]$ for all values of i and j. If

$$A = \begin{bmatrix} 10 & 15 \\ 20 & 14 \end{bmatrix} \qquad B = \begin{bmatrix} 21 & 13 \\ 12 & 17 \end{bmatrix}$$

Then

$$\begin{aligned} A + B &= \begin{bmatrix} 10 & 15 \\ 20 & 14 \end{bmatrix} + \begin{bmatrix} 21 & 13 \\ 12 & 17 \end{bmatrix} \\ &= \begin{bmatrix} (10 + 21) & (15 + 13) \\ (20 + 12) & (14 + 17) \end{bmatrix} = \begin{bmatrix} 31 & 28 \\ 32 & 31 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} A - B &= \begin{bmatrix} 10 & 15 \\ 20 & 14 \end{bmatrix} - \begin{bmatrix} 21 & 13 \\ 12 & 17 \end{bmatrix} \\ &= \begin{bmatrix} (10 - 21) & (15 - 13) \\ (20 - 12) & (14 - 17) \end{bmatrix} = \begin{bmatrix} -11 & 2 \\ 8 & -3 \end{bmatrix} \end{aligned}$$

If we have:

$$\begin{bmatrix} 10 & 20 \\ 50 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 20 \\ 50 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then the result is a *zero matrix* (which performs the same function as zero in ordinary arithmetic).

Addition of matrices is said to be *commutative* (sequence makes no difference) since $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

EXAMPLE

The company, Comfy Chairs Ltd, have presented proposed weekly output from a new factory in matrix **E** shown below.

$$\mathrm{E} = \begin{bmatrix} 30 & 30 & 0 \\ 18 & 20 & 0 \end{bmatrix}$$

Again, columns represent chair types and rows represent wood used. What would the combined weekly output be from the existing and new factories?

$$\mathrm{D} + \mathrm{E} = \begin{bmatrix} 30 & 60 & 80 \\ 20 & 30 & 40 \end{bmatrix} + \begin{bmatrix} 30 & 30 & 0 \\ 18 & 12 & 0 \end{bmatrix} = \begin{bmatrix} 60 & 90 & 80 \\ 38 & 42 & 40 \end{bmatrix}$$

25.2.3 Scalar multiplication of a matrix

A matrix may be multiplied by a single number or *scalar*. To do this we multiply each element of the matrix by the scalar, e.g.

$$5 \times \begin{bmatrix} 4 & 8 & 3 \\ 17 & 2 & 12 \end{bmatrix} = \begin{bmatrix} 20 & 40 & 15 \\ 85 & 10 & 60 \end{bmatrix}$$

In terms of matrix notation, if $\mathbf{A}$ is the matrix and c is the scalar then $c\mathbf{A} = [ca_{ij}]$ for all values of i and j.

EXAMPLE

Suppose Comfy Chairs Ltd plan to increase output across the range from their existing factory by 10 per cent. To increase output across the range by 10 per cent, we need to multiply each element by 1.1.

$$\begin{aligned} 1.1 \times \mathrm{D} &= \begin{bmatrix} 1.1 \times 30 & 1.1 \times 60 & 1.1 \times 80 \\ 1.1 \times 20 & 1.1 \times 30 & 1.1 \times 40 \end{bmatrix} \\ &= \begin{bmatrix} 33 & 66 & 88 \\ 22 & 33 & 44 \end{bmatrix} \end{aligned}$$

It is often convenient to reverse this argument, and take a common factor out of the matrix (cf. taking a common factor out of a bracket), to make further calculations easier, e.g.

$$\begin{bmatrix} 10 & 170 & 100 \\ 20 & 90 & 95 \\ 140 & 30 & 50 \end{bmatrix} = 10 \times \begin{bmatrix} 1 & 17 & 10 \\ 2 & 9 & 9.5 \\ 14 & 3 & 5 \end{bmatrix}$$

When the *same* matrix is to be multiplied by a series of scalars, we have:

$$a\mathrm{A} + b\mathrm{A} + c\mathrm{A} + d\mathrm{A} = (a + b + c + d)\mathrm{A}$$

25.2.4 Multiplication of matrices

When two matrices are to be multiplied together it is first necessary to check that the multiplication is possible; the matrices must be *conformable for multiplication*. This condition is satisfied if the number of columns in the first matrix is equal to the number of rows in the second matrix. The outcome of multiplication will be a matrix with the same number of rows as the first matrix and the same number of columns as the second matrix. If matrix **A** is of order $(a \times b)$ and matrix **B** is of order $(c \times d)$, then for multiplication to be possible, b must equal c and the new matrix produced by the product **AB** will be of order $(a \times d)$.

Thus a (2×3) matrix multiplied by a (3×2) matrix will give a (2×2) result; whereas a (3×2) matrix multiplied by a (2×3) matrix will give a (3×3) result. It is not possible to multiply a matrix of order (4×8) by a matrix of order (5×3). Note that even though the resultant matrix may be of the same order, multiplication is *not commutative* since **AB** does not necessarily equal **BA**.

The process of multiplication involves using a particular row from the first matrix and a particular column from the second matrix; placing the result as a single element in the result matrix, e.g.

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 10 & 13 \\ 11 & 14 \\ 12 & 15 \end{bmatrix} \quad A \times B = C$$

To find **A** × **B**, we will work out each element separately. For example, taking the first row of **A** and the first column of **B** gives the element c_{11} in the first row and column of **C**, i.e.

$$\begin{bmatrix} 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = (1 \times 10) + (4 \times 11) + (7 \times 12) = 138$$

$$= c_{11}$$

Note that we have gone along the row of the first matrix and down the column of the second matrix, multiplying the corresponding elements.

To find the *second* element in the first row of **C**, we take the first row of the matrix **A**, and the *second* column of the matrix **B**.

$$\begin{bmatrix} 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} 13 \\ 14 \\ 15 \end{bmatrix} = (1 \times 13) + (4 \times 14) + (7 \times 15) = 174$$

$$= c_{12}$$

This process continues, using the second row from **A**, and then the third row. In general, the *m*th row of **A** by the *n*th column of **B** gives the element c_{mn} in the *m*th row and *n*th column of **C**.

EXERCISE

Calculate the remaining elements of the matrix **C** Answer:

$$C = \begin{bmatrix} 138 & 174 \\ 171 & 216 \\ 204 & 258 \end{bmatrix}$$

Note that it is not possible to find **B** × **A** because the number of columns in **B** is not the same as the number of rows in **A**.

EXAMPLE

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 18 & 23 \\ 31 & 39 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 19 \\ 19 & 50 \end{bmatrix}$$

This confirms that $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Since the order in which the matrices are multiplied together is crucial to the product obtained, it is useful to have a terminology to show this. In the product **A** × **B, A** is said to be *postmultiplied* by **B**, while **B** is said to be *premultiplied* by **A**.

We have seen, above, the zero matrix which performs the same function in matrix algebra as a zero in arithmetic. We also need a matrix which will perform the function that unity (one) plays in arithmetic (e.g. $5 \times 1 = 5$).

For any matrix **A**, what matrix has no effect when **A** is postmultiplied by it? i.e.

$$A \times ? = A$$

The matrix which performs this function is known as the *identity matrix* and consists of 1s on the diagonal from top left (a_{11}) to bottom right (a_{nn}) and 0s for all other elements. Note that all identity matrices are *square* and that the 1s are said to be on the leading or *principal diagonal*.

Thus:

$$\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 6 & 8 & 10 \\ 12 & 20 & 14 \\ 2 & 75 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 10 \\ 12 & 20 & 14 \\ 2 & 75 & 3 \end{bmatrix}$$

If **I** is used for the identity matrix, we have:

$$AI = IA = A$$

EXAMPLE

Suppose Comfy Chairs Ltd. decide not to implement an overall increase in output across the range by 10 per cent from their existing factory but rather increase the Classic by 5 per cent, the Victorian by 10 per cent and the Modern by 15 per cent in both mahogany and teak.

This can be done by multiplying matrix **D** by a new matrix **F**:

$$\mathrm{DF} = \begin{bmatrix} 30 & 60 & 80 \\ 20 & 30 & 40 \end{bmatrix} \begin{bmatrix} 1.05 & 0 & 0 \\ 0 & 1.10 & 0 \\ 0 & 0 & 1.15 \end{bmatrix} = \begin{bmatrix} 31.5 & 66 & 92 \\ 21 & 33 & 46 \end{bmatrix}$$

EXAMPLE

Suppose we want to know the total number of chairs produced in mahogany or teak from the existing factory before any increase.

We can do this by multiplying matrix **D** by a column vector of 1s:

$$\begin{bmatrix} 30 & 60 & 80 \\ 20 & 30 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 170 \\ 90 \end{bmatrix}$$

25.2.5 Inverse of a matrix

As in arithmetic, division of one matrix by a second matrix is equivalent to multiplying the first by the *inverse* of the second, bearing in mind that the operation is not commutative. To transfer a matrix from one side of an equation to the other, both sides are multiplied by the inverse of the matrix. The inverse of a matrix **A** is denoted by $\mathbf{A}^{-1}$. A matrix multiplied in either order by the inverse of itself will always give the identity matrix, i.e.

$$AA^{-1} = A^{-1}A = I$$

Suppose we have:

$$A \times B = C$$

and we want to find **B**. If we *premultiply* both sides by: A^{-1}:

$$A^{-1}AB = A^{-1}C$$

$$IB = A^{-1}C$$

$$B = A^{-1}C$$

To find **A**, we can *postmultiply* both sides by $\mathbf{B}^{-1}$:

$$ABB^{-1} = CB^{-1}$$

$$AI = CB^{-1}$$
$$A = CB^{-1}$$

Finding the inverse of a matrix is *only* possible if the matrix is *square*, i.e. has the same number of rows as columns, and we shall initially look at the special case of a (2 × 2) matrix and then suggest a method for larger matrices.

The inverse of a (2 × 2) matrix

If

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{(a_{11}a_{22}) - (a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}$$

Putting this into words, the fraction in front of the matrix is one over the product of the two elements on the principal diagonal minus the product of the other two elements. Within the matrix the two elements on the principal diagonal change places and the other two elements change sign, e.g. if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

then

$$\begin{aligned} A^{-1} &= \frac{1}{(1 \times 4) - (2 \times 3)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ &= -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \end{aligned}$$

To check the answer we can find $\mathbf{AA}^{-1}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The denominator of the fraction we calculated above is called the *determinant* of the matrix, and if this is 0 then the matrix is said to be singular, and does not have an inverse.

Larger matrices

To find the inverse for a matrix larger than (2 × 2) is a somewhat more complex procedure and may be done by using the method of *cofactors* or by *row operations*. We shall give an example of how to find the inverse for a (3 × 3) matrix. (For larger matrices it is probably advisable to use a computer program.)

To find an inverse using the cofactors, we must first find the determinant of the matrix, denoted by $|\mathbf{A}|$. If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{22} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{22}a_{33} - a_{13}a_{32}) \\ &\quad + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \end{aligned}$$

A cofactor of an element consists of the determinant of those elements which are not in the same row and not in the same column as that element. Thus the cofactor of a_{11} is:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

and this can be evaluated as $(a_{22}a_{33}) - (a_{23}a_{32})$. To form the inverse matrix, each element of the initial matrix is replaced by its *signed* cofactor. (Note the signs of the cofactors replacing a_{12}, a_{21}, a_{23} and a_{32})

$$\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

After each element of this new matrix has been evaluated, it is *transposed*, that is, each column is written as a row, so that a_{12} becomes a_{21}. The resultant matrix is then multiplied by one over the determinant. This is likely to become considerably clearer as we work through an example.

EXAMPLE

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$$

Then

$$\begin{aligned} |\mathbf{A}| &= 1 \times (3 \times 12 - 5 \times 5) - 1 \times (2 \times 12 - 3 \times 5) + 1 \times (2 \times 5 - 3 \times 3) \\ &= 11 - 9 + 1 = 3 \end{aligned}$$

Replacing elements by cofactors, we have:

$$\begin{bmatrix} \begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

EXAMPLE

Transposing this, we have:

$$= \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

This is called the *adjunct matrix*, and $\mathbf{A}^{-1}$ is this matrix multiplied by the reciprocal of the determinant.

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

EXERCISE

Check that $\mathbf{AA}^{-1} = \mathbf{I}$:

To find the inverse using row operations, we create a *partitioned matrix*, by putting an identity matrix alongside the original matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 5 & 12 & 0 & 0 & 1 \end{array}\right]$$

Our objective now is to multiply and divide each row, or add and subtract rows until the matrix on the *left* of the partition is an identity. At that point, whatever is to the *right* of the partition will be the inverse of the original matrix. We already have a 1 at a_{11}, so to change the 1 at a_{21} to 0 we may subtract row 1 from row 2. (Note that we subtract corresponding elements for the *whole* row.)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 1 & 5 & 12 & 0 & 0 & 1 \end{array}\right]$$

To alter the 1 to a 0 at a_{31}, we again subtract row 1 from row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 9 & -1 & 0 & 1 \end{array}\right]$$

To alter the 2 to a 0 at a_{12}, we can subtract *two* times row 2 from row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 9 & -1 & 0 & 1 \end{array}\right]$$

To alter the 3 to a 0 at a_{32}, subtract *three* times row 2 from row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -3 & 1 \end{array}\right]$$

To reduce the 3 to a 1 at a_{33}, divide row 3 by 3.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -1 & 1/3 \end{array}\right]$$

To alter the -1 to a 0 at a_{13}, add row 3 to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/3 & -3 & 1/3 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & -1 & 1/3 \end{array}\right]$$

To alter the 2 to a 0 at a_{23}, subtract *two* times row 3 from row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/3 & -3 & 1/3 \\ 0 & 1 & 0 & -7/3 & 3 & -2/3 \\ 0 & 0 & 1 & 2/3 & -1 & 1/3 \end{array}\right]$$

As you will see, this is the same answer that was achieved by using the cofactors method. Note also that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ as you can easily prove.

25.3 Solutions of simultaneous equations

Matrices provide a methodology to manage sets of equations. Most of the equations encountered at this level will give unique solutions, e.g. $x = 5$, but this is not always the case. The single equation $x + 2y = 10$ does not have a unique solution but a number of solutions, e.g. $x = 0$ and $y = 5$ *or* $x = 2$ and $y = 4$ *or* $x = -10$ and $y = 10$ etc. In general, we need the same number of independent equations as variables to obtain unique solutions.

25.3.1 Solving equations using matrix inverses

Consider the equations:

$$7x + 4y = 80 \qquad 5x + 3y = 58$$

This pair of equations may be written in matrix notation as:

$$\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 80 \\ 58 \end{bmatrix}$$

or

$$\mathrm{Ax} = \mathrm{b}$$

If we premultiply by $\mathbf{A}^{-1}$ we have

$$\mathrm{A}^{-1}\mathrm{Ax} = \mathrm{A}^{-1}\mathrm{b}$$
$$\mathrm{Ix} = \mathrm{A}^{-1}\mathrm{b}$$
$$\mathrm{x} = \mathrm{A}^{-1}\mathrm{b}$$

The result will hold for *any* set of simultaneous equations where there are as many equations as unknowns; and thus if we premultiply the vector **b** by the inverse of the matrix **A**, we shall be able to find the values of the unknowns that satisfy the equations. Here:

$$\mathrm{A}^{-1} = \frac{1}{21 - 20}\begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$$

Therefore,

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 80 \\ 58 \end{bmatrix} \\ &= \begin{bmatrix} 240 - 232 \\ -400 + 406 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 6 \end{bmatrix}\end{aligned}$$

Thus $x = 8$ and $y = 6$.
If we have three equations:

$$\begin{aligned} 4x_1 + 3x_2 + x_3 &= 8 \\ 2x_1 + x_2 + 4x_3 &= -4 \\ 3x_1 + x_3 &= 1 \end{aligned}$$

then it can be shown that the inverse of the **A** matrix is:

$$\mathrm{A}^{-1} = \frac{1}{31}\begin{bmatrix} 1 & -3 & 11 \\ 10 & 1 & -14 \\ -3 & 9 & -2 \end{bmatrix}$$

And that the solution to the equations is:

$$\begin{aligned}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \frac{1}{31}\begin{bmatrix} 1 & -3 & 11 \\ 10 & 1 & -14 \\ -3 & 9 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} \\ &= \frac{1}{31}\begin{bmatrix} 31 \\ 62 \\ -62 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\end{aligned}$$

thus $x_1 = 1$, $x_2 = 2$ and $x_3 = -2$

25.3.2 Solving equations using Cramer's rule

Consider the following set of equations:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

We need to find the determinant of the matrix of coefficients, **A**, and the determinant of this matrix with each column replaced in turn by the column vector of constant terms.

The determinant of matrix **A** may be written

$$|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and the determinants of the matrix with column replacement

$$|\mathbf{A}_1| = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \qquad |\mathbf{A}_2| = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix} \qquad |\mathbf{A}_3| = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$$

If $|\mathbf{A}| \neq 0$, the unique solution is given by:

$$X = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} \qquad Y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} \qquad Z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|}$$

EXAMPLE

Given

$$\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} 4 & 3 & 1 \\ 2 & 1 & 4 \\ 3 & 0 & 1 \end{vmatrix} = 31 \qquad |\mathbf{A}_1| = \begin{vmatrix} 8 & 3 & 1 \\ -4 & 1 & 4 \\ 1 & 0 & 1 \end{vmatrix} = 31$$

$$|\mathbf{A}_2| = \begin{vmatrix} 4 & 8 & 1 \\ 2 & -4 & 4 \\ 3 & 1 & 1 \end{vmatrix} = 62 \qquad |\mathbf{A}_3| = \begin{vmatrix} 4 & 3 & 8 \\ 2 & 1 & -4 \\ 3 & 0 & 1 \end{vmatrix} = -62$$

Therefore

$$x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{31}{31} = 1$$

$$x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{62}{31} = 2$$

$$x_3 = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{-62}{31} = -2$$

25.3.3 Equations without unique solutions

Not all sets of simultaneous equations will give unique solutions like $x_1 = 1$, $x_2 = 2$ and $x_3 = -2$. Consider the following.

$$\begin{aligned} 2x + y - z &= 9 \\ x - y + 2z &= 5 \\ 4x + 2y - 2z &= 18 \end{aligned}$$

This can be written:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 18 \end{bmatrix}$$

Using the notation $\mathbf{Ax} = \mathbf{b}$, we could attempt to find the inverse of $\mathbf{A}$ by the partitioned matrix method.

However, we cannot obtain an identity matrix on the left hand side. In this case we cannot obtain a unique solution, because the equations are not independent. Looking at the original set of equations, it can be seen that the third equation was double the first. We only have two independent equations and three variables.

If we now consider only the first two independent equations, and take twice equation 1 from equation 2 we can eliminate z:

$$5x + y = 23$$

At this stage we can identify combinations of x and y that satisfy this condition, but not unique solutions.

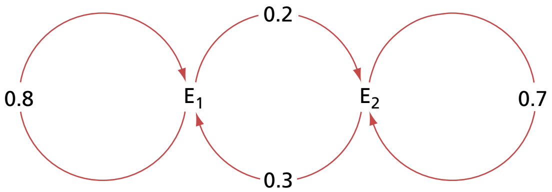
25.4 Markov chains

A Markov chain combines the ideas of probability with those of matrix algebra as discussed above. We will present this section as if you are already familiar with probability in order to keep the continuity of the concept, but you should refer to chapter 8 if that is not so. The Markov chain concept assumes that probabilities remain fixed over time, but that the system that is being modelled is able to change from one state to another, using these fixed values as *transition probabilities*. Consider, for example, the following transition matrix:

$$P = \begin{array}{c} \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

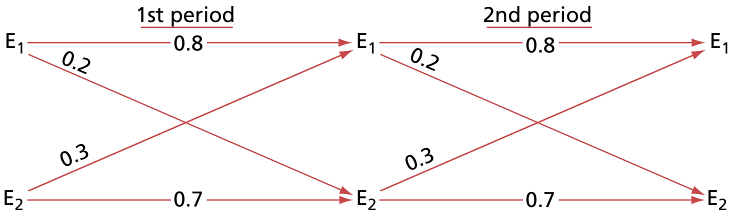
This means that if the system is in some state labelled E_1, the probability of going to E_2 is 0.2. If the system is at E_2, then the probability of going to E_1 is 0.3, and the probability of remaining at E_2 is 0.7. This transition matrix could be represented by the *directed graph* (Figure 25.1).

Figure 25.1



If we consider the movement from one state to another to happen at the end of some specific time period, and look at the passage of two of these periods, we have a situation as shown in Figure 25.2.

Figure 25.2



The probability of ending at E_1 after two periods if the system started at E_1 will be:

$$P(E_1 \rightarrow E_1 \rightarrow E_1) + P(E_1 \rightarrow E_2 \rightarrow E_1) = (0.8)(0.8) + (0.2)(0.3) = 0.70$$

Starting at E_1 and ending at E_2:

$$P(E_1 \rightarrow E_1 \rightarrow E_2) + P(E_1 \rightarrow E_2 \rightarrow E_2) = (0.8)(0.2) + (0.2)(0.7) = 0.30$$

Starting at E_2 and ending at E_1:

$$P(E_2 \rightarrow E_1 \rightarrow E_1) + P(E_2 \rightarrow E_2 \rightarrow E_1) = (0.3)(0.8) + (0.7)(0.3) = 0.45$$

Starting at E_2 and ending at E_2:

$$P(E_2 \rightarrow E_2 \rightarrow E_2) + P(E_2 \rightarrow E_1 \rightarrow E_2) = (0.7)(0.7) + (0.3)(0.2) = 0.55$$

Thus the transition matrix for *two* periods will be:

$$\begin{array}{c c} & \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \end{array} & \begin{bmatrix} 0.70 & 0.30 \\ 0.45 & 0.55 \end{bmatrix} \end{array}$$

but note that this is equal to $\mathbf{P}^2$, i.e. the square of the transition matrix for one period. To find the transition matrix for four periods, we would find $\mathbf{P}^4$ and so on.

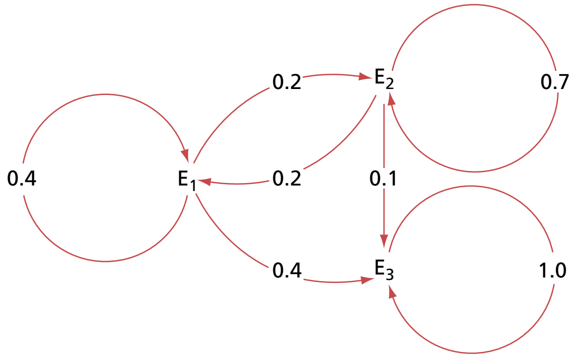
The states of the system at a given instant could be an item working or not working, a company being profitable or making a loss, an individual being given a particular promotion or failing at the interview, etc. In all transition matrices, the movement over time is from the state on the left to the state above the particular column, and thus, since something must happen, the sum of any row must be equal to 1.

A state is said to be *absorbent* if it has a probability of 1 of returning to itself each time. In the matrix

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

the state E_3 is an absorbent state, since each time the system reaches state E_3 it remains there. This can again be shown by a directed graph (Figure 25.3).

Figure 25.3



To use these transition matrices for predicting a future state, we need to know the initial state, which is written in the form of a vector. For example, if state E_1 in Figure 25.1 were a company being profitable, and E_2 were a company not being profitable, then we would consider what would happen to a group of companies. If the group consists of 150 profitable companies and 50 non-profitable companies, the initial vector will be

$$\begin{array}{cc} & \begin{array}{cc} E_1 & E_2 \end{array} \\ \mathrm{A}_0 = & [150 \quad 50] \end{array}$$

To find the situation after one time period, say one year, we postmultiply the initial state vector by the transition matrix:

$$[150 \quad 50]\begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = [135 \quad 65] = \mathrm{A}_1$$

$\mathbf{A}_1$ now represents the situation after one time period, where we would expect 135 companies to be profitable, and 65 not to be profitable. To find the situation after two time periods, we either multiply the initial state vector by P^2

$$[150 \quad 50]\begin{bmatrix} 0.7 & 0.3 \\ 0.45 & 0.55 \end{bmatrix} = [127.5 \quad 72.5] = \mathrm{A}_2$$

or postmultiply the vector $\mathbf{A}_1$ by P:

$$[135 \quad 60]\begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = [127.5 \quad 72.5] = \mathrm{A}_2$$

Both calculations give a row vector labelled A_2, which represents the expected number of companies in each state after two years. Note that some of the companies that are currently profitable may not have been profitable after only one year.

The process given above can continue with any transition matrix for any number of time periods; however, it is likely that the probabilities within the matrix will become out of date, and thus not fixed, if a prediction into the distant future is made. Markov chains are often used in manpower planning exercises and in market predictions.

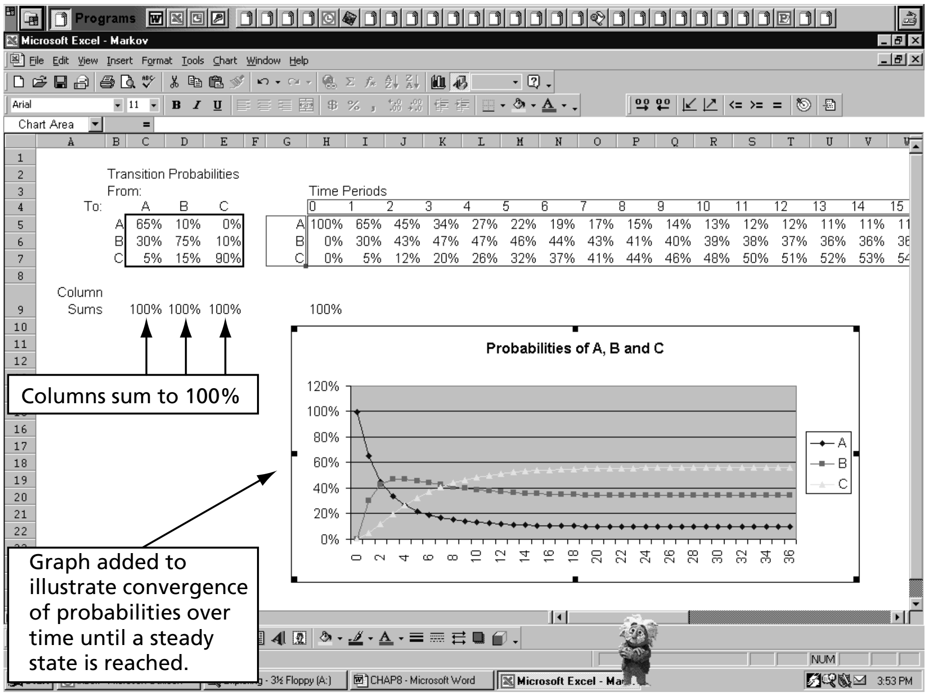
Multiplying out matrices can be a very time consuming process and offers many opportunites for mistakes. Fortunately, we can use various programs and packages to do the multiplication for us.

INSIGHT: To use this package, start EXCEL in the normal way and then load MARKOV.XLS from the website. (you will have to install INSIGHT the first time you use it) You should see a transition matrix as follows:

$$\begin{array}{cccc} & A & B & C \\ A & 65\% & 10\% & 0\% \\ B & 30\% & 75\% & 10\% \\ C & 5\% & 15\% & 90\% \end{array}$$

And you should also notice that the matrix is arranged so that the columns add to 100 per cent. To the right of the transition matrix is a table which shows the situation after 1, 2, 3, etc. time periods. Looking across this table, you can see that convergence occurs fairly quickly in this case. (Note that time period 0 is the initial state.) Clicking on the tab at the bottom of the screen will allow you to see a graph of this table.

This basic spreadsheet (figure 25.4) will allow you to solve any Markov chain problem. For transition matrices which are 2 × 2, then make the C row and C column equal to zeros. For larger transition matrices, insert new rows and columns as necessary. (Note: you should not



Transition Probabilities

To:	From: A	B	C
A	65%	10%	0%
B	30%	75%	10%
C	5%	15%	90%
Column Sums	100%	100%	100%

Time Periods

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	100%	65%	45%	34%	27%	22%	19%	17%	15%	14%	13%	12%	12%	11%	11%	
B	0%	30%	43%	47%	47%	46%	44%	43%	41%	40%	39%	38%	37%	36%	36%	
C	0%	5%	12%	20%	26%	32%	37%	41%	44%	46%	48%	50%	51%	52%	53%	
	100%															

Columns sum to 100%

Graph added to illustrate convergence of probabilities over time until a steady state is reached.

Figure 25.4 Markov chain using INSIGHT insert over the A column or row.) You will also need to increase the number of rows in the table to the right, and copy the formulae down the columns, and then across the table.

25.5 Leontief input–output analysis

This input–output model was first developed by Leontief in the 1940s. It recognized that industries or economic sectors do not produce in isolation but rely on each other. The output from one sector may be the input to the same sector, or to another productive sector, or may be consumed as final demand. Final demand refers to the non-productive sector, including households and exports. The agricultural sector, for example, will retain some of its output as seed or breeding stock, transfer the majority of its output to the food processing industries with the remainder of its output going directly for home consumption or exports.

In a complex economy, the segregation may be fairly arbitrary since many companies produce a range of products, that may fall into several sectors, but in the UK and the rest of Europe, the segregation is still attempted by using a Standard Industrial Classification of about 40 sectors. We will not attempt to model a complete system of 40 sectors (!) but will analyse a hypothetical economy with three productive sectors. The same approach, however, could be applied to a larger system.

If the three productive sectors have outputs x_1, x_2 and x_3, part of this output will go to other productive sectors and part to final demand. An important feature of this model is that for each sector, total output is equal to total input. An example is given in Table 25.1.

Table 25.1

Outputs from:	sector X_1	Inputs to: sector X_2	sector X_3	Final demand	Total output
X_1	20	10	60	110	200
X_2	40	10	50	150	250
X_3	10	60	30	400	500
Other inputs	130	170	360		
	200	250	500		

The first three columns show the inputs to sectors X_1, X_2 and X_3. It can be seen in the first column that sector X_1 uses 20 units of its own output, 40 units from sector X_2 and 10 units from sector X_3. The remaining 130 units are referred to as **primary inputs** and include items such as labour and raw materials. The fourth column gives final demand which is the difference between total output and output used by other productive sectors. A **control economy** would attempt to manage these final demand levels through the planning of industry or sectoral output. Clearly if high proportions of output are destined for other industries, the amount available for final demand will be reduced.

To produce 200 units from sector X_1, we need inputs of 20, 40 and 10 from sectors X_1, X_2 and X_3 respectively. Expressed as proportions we get 0.10, 0.20 and 0.05 which are known as the **input–output coefficients** or **technical coefficients**. Another important feature of the model is that these input–output coefficients remain *constant* regardless of changes in final demand. These input–output coefficients are placed into a matrix **A**, and using matrix notation,

$$a_{11} = 0.10 \qquad a_{21} = 0.20 \qquad a_{31} = 0.05$$

In general a_{ij} is the proportion of input from X_i in the output of X_j. The matrix **A** is known as the **input–output matrix** and can be written:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0.10 & 0.04 & 0.12 \\ 0.20 & 0.04 & 0.10 \\ 0.05 & 0.24 & 0.06 \end{bmatrix}$$

The input–output system shown in Table 24.1 can be represented by matrices:

$$\begin{bmatrix} 0.10 & 0.04 & 0.12 \\ 0.20 & 0.04 & 0.10 \\ 0.05 & 0.24 & 0.06 \end{bmatrix} \begin{bmatrix} 200 \\ 250 \\ 500 \end{bmatrix} + \begin{bmatrix} 110 \\ 150 \\ 400 \end{bmatrix} = \begin{bmatrix} 200 \\ 250 \\ 500 \end{bmatrix}$$

In summary:

$$\mathbf{AX} + \mathbf{D} = \mathbf{X}$$

where **X** is the output matrix and **D** is the demand matrix. This equation is known as the input–output equation. To find outputs to meet any projected final demands we must solve the above equation in terms of **X**. Using matrix algebra:

$$\begin{aligned} \mathbf{X} - \mathbf{AX} &= \mathbf{D} \\ (\mathbf{I} - \mathbf{A})\mathbf{X} &= \mathbf{D} \\ (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})\mathbf{X} &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} \\ \mathbf{X} &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} \end{aligned}$$

Taking matrix **A** away from the identity matrix **I** gives

$$(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} 0.90 & -0.04 & -0.12 \\ -0.20 & 0.96 & -0.10 \\ -0.05 & -0.24 & 0.94 \end{bmatrix}$$

The inverse of **(I − A)** is

$$\frac{10}{77\,132}\begin{bmatrix} 8784 & 664 & 1192 \\ 1930 & 8400 & 1140 \\ 960 & 2180 & 8560 \end{bmatrix}$$

Having developed the input–output model, we are now told that the final demand levels required from X_1, X_2 and X_3 are 77 132, 231 396 and 385 660 respectively. To achieve these levels we can solve the input–output equation in terms of **X**, to determine industry outputs.

EXERCISE

Using the figures from the example show that

$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{D}$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{10}{77\,132}\begin{bmatrix} 8784 & 664 & 1192 \\ 1930 & 8400 & 1140 \\ 960 & 2180 & 8560 \end{bmatrix}\begin{bmatrix} 77\,132 \\ 231\,396 \\ 385\,660 \end{bmatrix}$$

$$= \begin{bmatrix} 167\,360 \\ 328\,300 \\ 503\,000 \end{bmatrix}$$

The outputs required from sectors X_1, X_2 and X_3 are 167 360, 328 300 and 503 000.

The equation system can be used to show the relationships between industries within an economy and the effect on final demand of various output and input changes. It was used in the UK in the early 1970s to look at the effects of a reduction of oil supplies on various industrial sectors. However, the difficulty with this system is not only its size for a complex economy but the implicit assumption that the input–output coefficients remain constant. This is unlikely to remain true as technical progress and innovation change the ways in which some industries operate. To take this into account, a new matrix of input–output coefficients must be derived every few years and this can be a costly and time-consuming process.

The measurement of inputs/outputs can also present problems. It is generally assumed that we can express all inputs and outputs in the same units, say millions of pounds. Clearly the outputs from some sectors are more tangible than others. In a model of the economy, how do we measure the output from the public sector?

25.6 Conclusions

In the last few pages we have introduced the concept of a matrix and defined a few relationships.

25.7 Questions

$$A = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 7 \\ 8 & 10 \end{bmatrix} \quad E = \begin{bmatrix} 4 & 12 & 8 \\ 7 & 2 & 5 \\ 9 & 1 & 3 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 2 & 10 \\ 1 & 0 & 2 \\ 1 & 7 & 1 \end{bmatrix}$$

$$G = \frac{1}{58}\begin{bmatrix} -14 & 68 & 4 \\ 1 & -9 & 8 \\ 9 & -5 & -2 \end{bmatrix} \quad H = \begin{bmatrix} 10 & 5 & 8 \end{bmatrix}$$

Using the matrices given above, find:

1 C + D

2 4A

3 AB

4 CD

5 $\mathbf{DC}$

6 $\mathbf{BC}$

7 $\mathbf{E} + \mathbf{F}$

8 $\mathbf{AE}$

9 $\mathbf{EF}$

10 $\mathbf{FB}$

11 $\mathbf{C}^{-1}$

12 $\mathbf{E}^{-1}$

13 $(\mathbf{AB})^{-1}$

14 $\mathbf{BA}$

15 $\mathbf{FG}$

16 $\mathbf{GF}$

17 $\mathbf{HF}$

18 $\mathbf{BH}$

19 $\mathbf{HB}$

20 $\mathbf{BHF}$

Solve the following sets of simultaneous equations using matrix algebra:

21
$$\begin{aligned} 4x + 2y &= 11 \\ 3x + 4y &= 9 \end{aligned}$$

22
$$\begin{aligned} 6x + 3y &= 18 \\ 2x - 3y &= 14 \end{aligned}$$

23
$$\begin{aligned} 5x - 3y &= 26 \\ 2x + 2y &= 4 \end{aligned}$$

24
$$\begin{aligned} 10x + 10y &= 6 \\ 3x + 7y &= 11 \end{aligned}$$

25
$$\begin{aligned} 4a + 3b + c &= 15 \\ 2a + 5b + 7c &= 47 \\ 5a + 6b - 2c &= 7 \end{aligned}$$

26
$$\begin{aligned} 10a + 3b + 4c &= 8 \\ 20a - 5b + 2c &= 12 \\ 25a - b + 5c &= 16 \end{aligned}$$

Using Cramer's rule, solve

27
$$\begin{aligned} 4x + 2y - z &= 57 \\ x - y + 2z &= -12 \\ 2x + 2y + z &= 31 \end{aligned}$$

28
$$\begin{aligned} 2a + b + c &= 7 \\ b + 3c &= 9 \\ 4a - b &= 1 \end{aligned}$$

Using a computer-based method, solve

29
$$\begin{aligned} a + 2b + c + 2d &= 5 \\ 2a + b + c - d &= 1 \\ 3a + b + 3c + d &= 10 \\ a - b - c + 2d &= 4 \end{aligned}$$

30
$$\begin{aligned} 4x_1 + 2x_2 - x_3 + x_4 &= 29 \\ x_1 + x_2 + x_3 - x_4 &= 7 \\ 2x_1 + 3x_2 + x_4 &= 20 \\ 3x_1 + x_2 - x_3 + 2x_4 &= 22 \end{aligned}$$

Using the notation developed in the previous section, analyse the following input–output tables:

31

Outputs from	Inputs to:		Final demand	Total output
	A	B		
A	10	40	50	100
B	30	20	50	100
Other	60	40		
	100	100		200

(a) Identify **A, X, D** and verify that the relationship holds.

(b) If final demand changes to 220 for *A* and 170 for *B*, find the level of output required from each sector.

(c) Reconstruct the table above given these new final demands.

32

Outputs from	Inputs to:		Final demand	Total output
	A	B		
A	5	85	10	100
B	20	120	360	500
Other	75	295		
	100	100		600

(a) Verify that the relationship between **A, X** and **D** holds for this table.

(b) Find the outputs necessary from each sector if the final demand changes to 50 for *A* and 500 for *B*.

33

Outputs from	Inputs to:			Final demand	Total output
	A	B	C		
A	20	10	30	40	100
B	20	40	60	80	200
C	20	20	120	140	300
Other	40	130	90		
	100	200	300		600

(a) Find the matrix of technical coefficients.

(b) Find $(\mathbf{I} - \mathbf{A})$.

(c) Show that the inverse of $(\mathbf{I} - \mathbf{A})$ is:

$$\frac{10}{342}\begin{bmatrix} 46 & 4 & 9 \\ 16 & 46 & 18 \\ 18 & 9 & 63 \end{bmatrix}$$

(d) Confirm that $\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{D}$.

(e) If final demand now changes to 342 for *A*, 684 for *B* and 1026 for *C*, find the levels of output necessary for each sector.

34

Outputs from	Inputs to:			Final demand	Total output
	A	B	C		
A	10	10	10	170	200
B	10	20	20	350	400
C	100	300	70	530	1000
Other	80	70	900		
	200	400	1000		1600

Using the information given above, find the levels of output necessary from each sector to support final demand figures of 1000 for *A*, 1000 for *B* and 400 for *C*.

35 Draw a directed graph for each of the transition matrices given below:

(a)

$$\begin{array}{c|cc} & E_1 & E_2 \\ \hline E_1 & 0.7 & 0.3 \\ E_2 & 0.5 & 0.5 \end{array}$$

(b)

$$\begin{array}{c|ccc} & E_1 & E_2 & E_3 \\ \hline E_1 & 1.0 & 0 & 0 \\ E_2 & 0.1 & 0.8 & 0.1 \\ E_3 & 0.1 & 0.8 & 0.1 \end{array}$$

In case (b) what will eventually happen to the system?

36 For the transition matrix $\mathbf{P}$, given below, find $\mathbf{P}^2$, $\mathbf{P}^4$, $\mathbf{P}^8$.

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.8 \\ 0 & 1.0 \end{bmatrix}$$

37 A particular market has 100 small firms, 50 medium-sized firms and 10 large firms, and it has been noticed that the transition matrix from year to year is represented by the matrix given below:

$$\begin{array}{ll}
 & \text{Size next period} \\
\text{Size this period} & \begin{array}{cccc} \text{Small} & \text{Medium} & \text{Large} & \text{Bankrupt} \end{array} \\
\begin{array}{l} \text{Small} \\ \text{Medium} \\ \text{Large} \\ \text{Bankrupt} \end{array} & \left[\begin{array}{cccc} 0.7 & 0.2 & 0 & 0.1 \\ 0.3 & 0.5 & 0.1 & 0.1 \\ 0 & 0.3 & 0.6 & 0.1 \\ 0.1 & 0 & 0 & 0.9 \end{array}\right]
\end{array}$$

No firms are bankrupt initially.

Find the expected number of firms in each category at the end of:

(a) 1 year
(b) 2 years
(c) 3 years
(d) 4 years.

38 A firm has five levels of intake of staff, and wishes to predict the way in which staff will progress through the various grades. Data has been collected to allow the construction of a transition matrix, including those who leave the firm. This is shown below:

$$\begin{array}{rc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \text{Left} \end{array} \\
\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \text{Left} \end{array} & \left[\begin{array}{cccccc} 0.3 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0 & 0.3 & 0.2 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{array}\right]
\end{array}$$

Six hundred and fifty people join in a particular year, at grades of 1–5 as set out below:

$$[200 \quad 150 \quad 150 \quad 100 \quad 50]$$

Use this information to predict:

(a) the numbers in each grade after one year
(b) the numbers in each grade after four years.

The wages for each grade from 1 to 5 are given below:

$$[50 \quad 60 \quad 80 \quad 100 \quad 120]$$

Find the total wage bill of the company for this cohort:

(c) as soon as they join
(d) after 4 years with the company (assuming the same wage levels).

39 Within a company, an individual's probability of being promoted depends on whether or not they were promoted in the previous year. They may also be made redundant. This situation may be modelled by a Markov process, with the following transition matrix:

$$\begin{array}{lc} & \text{This period} \\ \textit{Last period} & \begin{array}{ccc}\textit{Same job} & \textit{Promotion} & \textit{Redundant}\end{array} \\ \begin{array}{l}\textit{Same job}\\ \textit{Promotion}\\ \textit{Redundant}\end{array} & \begin{bmatrix}0.7 & 0.2 & 0.1\\ 0.9 & 0 & 0.1\\ 0 & 0 & 1.0\end{bmatrix}\end{array}$$

If, in the last time period, 100 people retained the same job, 5 were promoted and none were made redundant, find the expected numbers in each category after:

(a) one period
(b) two periods
(c) three periods.

After three periods, the economy becomes more buoyant, and the threat of redundancy is lifted. The transition matrix now becomes:

$$\begin{array}{lc} & \begin{array}{cc}\textit{Same job} & \textit{Promotion}\end{array} \\ \begin{array}{l}\textit{Same job}\\ \textit{Promotion}\end{array} & \begin{bmatrix}0.7 & 0.3\\ 0.8 & 0.2\end{bmatrix}\end{array}$$

For those still remaining with the company from the original cohort, find the expected numbers in each category after:

(d) the next period, i.e. period 4
(e) the following period, i.e. period 5
(f) period 6.

If you have access to the software, try the following questions:

A1 Use the Excel spreadsheet C8freqdef.xls (on the website) to find the probability of a head after 20 trials. Record your answer. Press F9 and again note the answer. Continue until you have 20 answers, and then find the average of your answers. What does this tell you? [see Section 10.4].

A2 As a comparator, repeat the exercise A1, but note the answer after only five trials. How does your final answer differ from your previous one?

A3 If you have access to INSIGHT, use the TREE.XLS spreadsheet to suggest the rational decision for the following problem. A business believes that the market for its product is very buoyant and will expand. In order to take advantage of the expectation, the company has to decide whether or not to expand its production capabilities and has identified that it could expand by 30% at a cost of £10m, expand by 10% at a cost of £2m or not expand at all (zero cost). Market analysts suggest that the probability of an expansion of 40% is 0.1, of an expansion of 20% is 0.3, an expansion by 10% is 0.4, or else it will remain constant. The following table gives the likely increase in profit to the company for various combinations of company expansion and market growth:

Market Growth	Company growth 30%	10%	0%
+40%	+£50m	+£30m	+£4m
+20%	+£30m	+£30m	+£4m
+10%	+£12m	+£12m	+£4m
+0%	+£2m	+£2m	+£0m

A4 A new product can be launched, market tested, the design sold on (for £25m), or the company can delay the decision for a year. Given the following data, suggest what the rational decision would be.

Product is launched at a cost of £4m:

Outcome	Probability	Profit increase
Big success	0.1	+£100m
Success	0.3	+£30m
OK	0.4	+£5m
Fails	0.2	−£5m

Product is market tested at cost of £0.5m:

Outcome	Probability
Pass	0.8
Fail	0.2

The company can then decide whether to launch, wait a year (at a cost of £0.2m) or sell the design for £35m. If they launch (cost = £4m), then the following table applies:

Outcome	Probability	Profit increase
Big success	0.1	+£100m
Success	0.5	+£30m
OK	0.3	+£5m
Fails	0.1	−£5m

If they launch after a year (cost £4m), the following table applies:

Outcome	Probability	Profit increase
Big success	0.15	+£90m
Success	0.55	+£25m
OK	0.25	+£10m
Fails	0.05	−£5m

Were the company to wait for a year (cost = £0.2m), they could then sell the design for £30m or launch at a cost of £4m with the following probabilities and outcomes:

Outcome	Probability	Profit increase
Big success	0.1	+£90m
Success	0.4	+£25m
OK	0.4	+£10m
Fails	0.1	−£5m

A5 If you have access to INSIGHT, use the MARKOV program to solve the following Markov chain:

		From	
To	A	B	C
A	0.1	0.4	0.5
B	0.5	0.1	0.4
C	0.4	0.5	0.1

What is the position after four time periods? What will be the equilibrium position?

A6 If you have access to INSIGHT, use the MARKOV program to solve the following Markov Chain:

		From		
To	A	B	C	D
A	0.9	0	0	0
B	0	0.9	0.1	0
C	0.1	0.1	0.8	0
D	0	0	0.1	1.0

What is the position after four time periods? What will be the equilibrium position?